Problem 6.1.

1. Compute the following:
   (a) $6^{365} \bmod 7$,
   (b) $2^{981} \bmod 5$,
   (c) $3^{23} \cdot 5^{24} \bmod 16$,
   (d) $392 \cdot 26019 \bmod 13$.
2. Prove that for any integer x that is not divisible by 3, and any even integer n, we have $x^n \equiv 1 \bmod 3$.
3. Compute the remainder of $2^{761^{435}+97!}$ modulo 3.
4. Show that there is no number n which is congruent to 3 mod 4 and to 5 mod 8.
5. Let m, x and e be positive integers. Let $e = (b_k \dots b_0)_2$ be the binary representation of e (i.e., $b_i \in \{0,1\}$, and $e = \sum_{i=0}^{k} b_i 2^i$). Let $x_0 = x \bmod m$ and for each $i > 0$, let $x_i = x_{i-1}^2 \bmod m$. Show that
$$x^e \equiv \prod_{i=0}^{k} x_i^{b_i} \bmod m.$$
6. Use the previous question to compute $5^{59} \bmod 23$.

Problem 6.2.

(This problem mostly requires the same techniques of the previous one. You can solve this to practise more on modular arithmetic.)

1. Compute the following without using a calculator:
   (a) $37^{121} \bmod 7$,
   (b) $18^{243} \bmod 19$,
   (c) $3^{17!} \bmod 27$,
   (d) $460002 \cdot 25 \bmod 23$,
   (e) $111223344556677889975310024681379 \bmod 8$,
2. Compute $65363549000917 \bmod 9$.
3. Decide whether or not the multiplication
$$6453601 \cdot 23456 = 151975665056$$
is correct by reducing mod 9.
4. Prove that given a number represented in base 10 as $n = (d_k \dots d_1 d_0)_{10}$ (with digits $d_i \in \{0,1,\dots,9\}$), we have $n \equiv \sum_{i=0}^{k} (-1)^i d_i \bmod 11$.
5. Using the previous question, compute $9760145571116 \bmod 11$.
6. Given a number n represented in base 10, find a similar method to compute n modulo 1001, and use it to compute 4067007258442 modulo 1001.

7. Which of the following numbers are multiples of 11?

$$a = 67^{9876543210112277} + 21^{1231239875566} + 9$$
$$b = 109^{4648731230355} + 56^{65659313514739945} + 1$$
$$c = 36^{102498765} + 90^{907864310} + 7$$

Problem 6.3.

1. Let $x = 021395789400$. Perform (by hand) the Euclidean division of x by 97.
   Compute the two control digits MOD 97-10 for the telephone number $x_1 = 021\ 395\ 7894$.
2. Let $\tilde{x}_1$ be the number obtained by replacing 02 with 99 in x_1:
   $$\tilde{x}_1 = 991\ 395\ 7894$$
   Compare the control digits MOD 97-10 of x_1 and $\tilde{x}_1$.
3. More generally, let z be an integer whose decimal representation includes twice the digit 9 in consecutive positions. Let z' be the number obtained by replacing 99 with 02 in z. Show that $z - z'$ is a multiple of 97.
   If 99 was replaced by mistake with 02, can the control digits of MOD 97-10 detect this error?
4. Suppose a bank uses MOD 97-10 to ensure that messages from customers are transmitted to the bank without modifications. It is known that the bank encodes wire transfer orders in the format
   $$M = DDDDDDDDDD \| AAAAAAA \| CC$$
   where the first part (represented by the D's) is the ten-digit receiving account number, the second part (represented by the A's, padded on the left by zeros if necessary) is the amount to be transferred, and the final part (represented by the C's) is the MOD 97-10 control digits. Note that there are exactly 7 digits in the 'Amount' part; bigger transactions will require in-person confirmation. Recall that '$\|$' denotes the concatenation operator.
   You, a software engineer by day, malicious hacker by night, have somehow managed to find a way to modify up to two digits in such messages while they are being transmitted. You are unhappy with your salary of CHF $10'000$ a month from your day job. You intercept the following message
   $$M_{\text{salary}} = 1461319897001000010$$
   for the salary of March '25 from your employer to the bank. How much more money can you make the bank transfer to your account instead?